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ON SOLVING THREE INTERESTING PUZZLES

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Abstract:
The study of equations whose solutions were in integers was studied for several centuries. Traditionally such equations are known as Diophantine Equations. In this paper, I will pose three related puzzles whose solutions depend on solving a particular Diophantine equation. By providing method for solving that particular equation using continued fraction, I will solve all three puzzles simultaneously.

Key Words: Diophantine Equations, Right Triangle, Arithmetic Progression, Square – Triangular Numbers, Pronic Numbers

1. Introduction:

Diophantine Equations are equations whose solutions must be in integers. These equations were introduced by Greek mathematician Diophantus around third century CE. Indian mathematician Brahmagupta studied such equations in great depth and provided ways of generating infinitely many solutions through recursion. In this paper, I will introduce three interesting puzzles and solve all of them through a single Diophantine Equation.

2. Definitions:

2.1 The sum of first n natural numbers are called triangular numbers and the nth triangular number is given by $\frac{n(n+1)}{2}$ (1)

2.2 The product of consecutive positive integers are called Pronic numbers. The mth Pronic number is given by $m(m+1)$ (2)

From (1) and (2), we notice that Pronic numbers are exactly twice that of triangular numbers.

3. Introduction of Three Puzzles:

3.1 First Puzzle:

Describe the lengths of all possible right triangles whose legs are in Arithmetic Progression.

3.2 Second Puzzle:

Describe all possible positive integers which are both triangular and square numbers.

3.3 Third Puzzle

Describe all possible positive integers which are both triangular and pronic numbers.

4. Solutions to the Diophantine Equation:

In this section, I will provide the method of solving the Diophantine Equation $x^2 - 2y^2 = -1$ (3)

Equation (3) is also referred as Pell's Equation. In order to solve (3), we first notice that

$$\left(1-\sqrt{2}\right)\times\left(1+\sqrt{2}\right) = -1 \Rightarrow 1-\sqrt{2} = \frac{-1}{1+\sqrt{2}} = \frac{-1}{2-\left(1-\sqrt{2}\right)} = \frac{-1}{2+\cfrac{1}{2-\left(1-\sqrt{2}\right)}}$$

$$1-\sqrt{2} = \frac{-1}{2+\cfrac{1}{2+\cfrac{1}{2-\left(1-\sqrt{2}\right)}}} = \cdots = \frac{-1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}$$

$$\sqrt{2} = 1+\frac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}} \quad (4)$$

The successive convergents of the continued fraction expression (4) are given by

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \cdots \quad (5)$$

Among these, if we consider first, third, fifth, seventh, in general odd positioned convergent and consider numerator as *x*, denominator as *y*, then we notice that (*x*, *y*) are solutions to (3).

That is, $(x, y) = (1,1);(7,5);(41,29);(239,169);\ldots$ are solutions to $x^2 - 2y^2 = -1 \quad (6)$

Similarly from the second, fourth, sixth, in general even positioned convergent, we notice that $(x, y) = (3,2);(17,12);(99,70);(577,408);\ldots$ are solutions to $x^2 - 2y^2 = 1 \quad (7)$

5. Solving the First Puzzle:

If we suppose that the legs of some right triangle are in Arithmetic Progression say of the form *a*, *a* + *d* where *a*, *d* are some positive integers and if we assume *c* to be hypotenuse of the triangle then by Pythagoras Theorem we get $a^2 + (a+d)^2 = c^2 \quad (8)$. Simplifying this, we obtain a equation of the form $X^2 - 2Y^2 = -1 \quad (9)$ where $X = \frac{2a+d}{d}, Y = \frac{c}{d}$.

But (9) is identical to (3) except the change in variables. Hence from (6), we see that the solutions to $X^2 - 2Y^2 = -1$ are given by $(X,Y) = (1,1);(7,5);(41,29);(239,169);\ldots$

Now from $X = \frac{2a+d}{d}, Y = \frac{c}{d}$ we get $a = \frac{d}{2}(X-1), c = dY \quad (10)$

If $(X,Y) = (1,1)$ then we get *a* = 0, giving a degenerate right triangle. Hence we neglect this case. Considering the other possible values of *X*, *Y* we see that

$(X,Y) = (7,5) \Rightarrow a = 3d, c = 5d; (X,Y) = (41,29) \Rightarrow a = 20d, c = 29d; (X,Y) = (239,169) \Rightarrow a = 119d, c = 169d$

Thus the sides of required right triangle whose legs are in Arithmetic Progression would be of the form $(3d,4d,5d);(20d,21d,29d);(119d,120d,169d);\ldots \quad (11)$ where *d* is any positive integer. Extracting more solutions from (6), we can determine more right triangles whose sides are in Arithmetic Progression.

6. Solving the Second Puzzle:

If we need to determine numbers which are both square and triangular then from (1) we obtain $\frac{n(n+1)}{2} = m^2 \quad (12)$. Multiplying both sides by 8 and simplifying, (12) can be re-written as $X^2 - 2Y^2 = 1 \quad (13)$ where $X = 2n+1, Y = 2m$.

But from (7) we notice that $(X,Y) = (3,2);(17,12);(99,70);(577,408);\ldots$ are solutions to (13). Hence, $m = 1, 6, 35, 204, \ldots \quad (14)$.

Thus from (12), the square – triangular numbers are given by 1, 36, 1225, 41616, . . . (15).

7. Solving the Third Puzzle:

In order to determine numbers which are both triangular and pronic, then from (1) and (2), we obtain $\frac{n(n+1)}{2} = m(m+1) \quad (16)$.

Multiplying by 4 on both sides and simplifying (16), we get $X^2 - 2Y^2 = -1 \quad (17)$ where $X = 2n+1, Y = 2m+1$. But (17) is precisely equation (3) except for change of variables. Hence from (6), we see that $(X,Y) = (1,1);(7,5);(41,29);(239,169);\ldots$ are solutions of (17).

If $(X,Y) = (1,1)$ then we get $n = \frac{X-1}{2} = 0, m = \frac{Y-1}{2} = 0$. Since we require the solutions in positive integers, we can neglect this solution.

$(X,Y) = (7,5) \Rightarrow n = \frac{X-1}{2} = 3, m = \frac{Y-1}{2} = 2; (X,Y) = (41,29) \Rightarrow n = \frac{X-1}{2} = 20, m = \frac{Y-1}{2} = 14;$

$(X,Y) = (239,169) \Rightarrow n = \frac{X-1}{2} = 119, m = \frac{Y-1}{2} = 84; (X,Y) = (1393,985) \Rightarrow n = \frac{X-1}{2} = 696, m = \frac{Y-1}{2} = 492$

Thus, the positive integers which are both triangular and pronic are given by 6, 210, 7140, 242556, . . .

8. Conclusion:

In this paper, I have posed three interesting puzzles which are intimately connected with the Diophantine Equations $x^2 - 2y^2 = \pm 1$. In section 4 equations (6) and (7) provides infinitely many solutions to $x^2 - 2y^2 = -1, x^2 - 2y^2 = 1$ respectively. Using the solutions of $x^2 - 2y^2 = -1$, I had solved first and third puzzles whereas using the solutions of $x^2 - 2y^2 = 1$, I had solved the second puzzle completely. Since (6) and (7) contains infinitely many solutions, all the three puzzles have infinitely many solutions.

It is very interesting and exciting to note that seemingly very different puzzles are connected with same Diophantine Equation. This is one of the beautiful aspects of mathematics that completely unrelated concepts would get related to a particular concept and we can solve complex problem using the related simple problem. This paper provides one such example.

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