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ON SOLVING BRAHMAGUPTA'S PUZZLE

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Abstract:

The great Indian mathematician Brahmagupta once said a quote as a response for a question about "Who is a mathematician?". In this paper, I will introduce the quote by Brahmagupta which represent a puzzle and provide its solution in a way as elegant as possible. Usually the solutions to the equation posed by Brahmagupta were solved using recurrence relations but I had used continued fraction concept to arrive at the solutions.

Key Words: Pell's Equation, Square – Free, Continued Fraction, Convergents, Infinitely Many Solutions

1. Introduction:

During 628 CE, one of the great Indian mathematicians Brahmagupta published his important treatise titled Brahmasphuta Siddantha which is a magnificent collection of mathematical and astronomical facts. To this day, this work remains one of classic contributions to mathematics. Brahmagupta is remembered today for providing arithmetical laws by treating zero as a number in its own right. During one occasion when Brahmagupta was asked whom he would consider as a mathematician, he said: "A person, who can, within a year, solve $x^2 - 92y^2 = 1$ is a mathematician". In fact, Brahmagupta himself had provided ways to solve such type of equations. In this paper, however, I had solved the question posed by Brahmagupta in a novel way using continued fraction which is very different from traditional way using recurrence relations obtained from the initial solution.

2. Brahmagupta – Pell's Equation:

Equations of the form $x^2 - dy^2 = 1 \quad (1)$, where *d* is a square-free number are called Pell's Equation though the name attributed to English mathematician John Pell has nothing to do with such equations. The greatest contributor of mathematics Leonhard Euler by mistake mentioned the name Pell's Equation and it got stuck in mathematical world since then. But Indian mathematician Brahmagupta had discussed equations of the form (1) in detailed fashion and had provided recurrent solutions that can generate subsequent solutions from the initial solution. Mathematicians of later period like Second Bhaskara, Fermat, Lagrange, Legendre had provided detailed methods of solving (1) through various possible ways. In recent times, matrix methods were employed to solve Pell's equations. The solutions through matrix methods are just reformulation of the ideas presented by Brahmagupta around 14 centuries earlier. In this paper, I will present a novel method to solve the equation $x^2 - 92y^2 = 1 \quad (2)$ posed by Brahmagupta and call it as Brahmagupta puzzle in his honor.

3. Solving Brahmagupta's Puzzle:

In the attempt of solving the puzzle posed by Brahmagupta represented by equation (2) namely, $x^2 - 92y^2 = 1$, first we observe the fact that $92 = 23 \times 4$.

Hence instead of solving (2) it is enough to solve $x^2 - 23Y^2 = 1 \quad (3)$ were $Y = 2y$. We now observe the following computations

$$(5-\sqrt{23})\times(5+\sqrt{23}) = 2 \Rightarrow 5-\sqrt{23} = \frac{2}{5+\sqrt{23}} = \frac{2}{10-(5-\sqrt{23})}$$

$$5-\sqrt{23} = \cfrac{2}{10-\cfrac{2}{10-(5-\sqrt{23})}} = \cfrac{2}{10-\cfrac{2}{10-\cfrac{2}{10-(5-\sqrt{23})}}} = \cdots = \cfrac{2}{10-\cfrac{2}{10-\cfrac{2}{10-\cfrac{2}{10-\cdots}}}}$$

Hence we obtain

$$\sqrt{23} = 5 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cdots}}}} \quad (4)$$

If we now determine the successive convergents from the continued fraction (4) we get the following numbers

$$\frac{5}{1}, 5 - \frac{2}{10} = \frac{24}{5}, 5 - \cfrac{2}{10 - \cfrac{2}{10}} = \frac{235}{49}, 5 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10}}} = \frac{1151}{240}, 5 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10}}}} = \frac{11275}{2351},$$

$$5 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10}}}}} = \frac{55224}{11515}, 5 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10}}}}}} = \frac{540965}{112799},$$

$$5 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10 - \cfrac{2}{10}}}}}}} = \frac{2649601}{552480}, \cdots$$

From the above successive convergents, if we consider first, third, fifth, seventh convergents then we get

$$\frac{5}{1}, \frac{235}{49}, \frac{11275}{2351}, \frac{540965}{112799}, \cdots$$

If we consider numerators and denominators of above rational numbers as x and Y respectively then we notice that $(x, Y) = (5,1), (235, 49), (11275, 2351), (540965, 112799), \ldots$ are subsequent solutions of $x^2 - 23Y^2 = 2 \ (5)$.

Similarly, if we consider second, fourth, sixth, eighth successive convergents from the above computation then we get $\frac{24}{5}, \frac{1151}{240}, \frac{55224}{11515}, \frac{2649601}{552480}, \cdots$

If we consider numerators and denominators of these rational numbers as x and Y respectively then we notice that $(x, Y) = (24, 5), (1151, 240), (55224, 11515), (2649601, 552480), \ldots$ are subsequent solutions of equation (3) given by $x^2 - 23Y^2 = 1$.

Since $Y = 2y$, the solutions to equation (2) namely $x^2 - 92y^2 = 1$ are obtained from the solutions of $x^2 - 23Y^2 = 1$, whenever Y is even. We observe that Y is even for the pairs $(1151, 240), (2649601, 552480), \ldots$ From these pairs, we find that $y = 120, 276240, \ldots$

Therefore the solutions to the Brahmagupta puzzle $x^2 - 92y^2 = 1$ are given by

$$(x, y) = (1151, 120), (2649601, 276240), \ldots$$

Hence the least possible solution in positive integers to the Brahmagupta puzzle $x^2 - 92y^2 = 1$ is given by $x = 1151$, $y = 120$.

We further notice that each solution to $x^2 - 23Y^2 = 1$ can be generated by using the recurrence relations $x_{n+2} = 48x_{n+1} - x_n, Y_{n+2} = 48Y_{n+1} - Y_n; x_0 = 1, x_1 = 24, Y_0 = 0, Y_1 = 5 \ (6)$ were $n \geq 0$.

Using this, we find that the first few solutions of $x^2 - 23Y^2 = 1$ are given by

$$(x,Y) = (1,0),(24,5),(1151,240),(55224,11515),(2649601,552480),$$
$$(127125624,26507525),(6099380351,1271808720),...$$

Hence the first few non – trivial solutions to the Brahmagupta puzzle $x^2 - 92y^2 = 1$ are given by $(x,y) = (1151,120),(2649601,276240),(6099380351,635904360),...$ (7)

Thus using the successive continued fraction convergents given by (4) or by using (6), we can generate infinitely many solutions to the Brahmagupta puzzle $x^2 - 92y^2 = 1$.

4. Conclusion:

By introducing the wonderful quote by Brahmagupta which happens to be a particular Pell's equation of the form $x^2 - 92y^2 = 1$, I have obtained infinitely many solutions to this equation given by (7). For this, I had converted the actual equation to much simpler equation of the form $x^2 - 23Y^2 = 1$. Now to solve this reduced equation, I have made use of the continued fraction computation for $\sqrt{23}$ as presented in (4). The continued fraction expansion in (4) provides alternate solutions to the two Pell's equations namely $x^2 - 23Y^2 = 2$ and $x^2 - 23Y^2 = 1$. Since we need to obtain integer solutions, the solutions to $x^2 - 23Y^2 = 1$ when Y is even provides the solutions to the actual equation $x^2 - 92y^2 = 1$. Thus the paper discuss about the novelty in solving the puzzle posed by Brahmagupta. Of course, for doing this, it is enough to consume few productive hours in a day. So in today's scenario, the quote by Brahmagupta can be rephrased as "A person, who can, within a day, solve $x^2 - 92y^2 = 1$ is a mathematician".

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